

ON PROOFS OF CERTAIN COMBINATORIAL IDENTITIES

GEORGE GROSSMAN, AKALU TEFERA AND AKLILU ZELEKE

ABSTRACT

In this paper we formulate combinatorial identities that give representation of positive integers as linear combination of even powers of 2 with binomial coefficients. We present side by side combinatorial as well as computer generated proofs using the Wilf-Zeilberger(WZ) method.

1. INTRODUCTION

It is known that every integer can be written as a sum of integral powers of 2. A somewhat related problem is to find for every positive integer n a positive integer k depending on n with $k(n) < k(n+1)$ and integer coefficients $a_i, i = 0, 1, \dots, k-1$ such that

$$(1) \quad n = \sum_{i=0}^{k-1} a_i 2^{2^i}.$$

The background and motivation for this problem lies in studying the zeros of the j -th order polynomial of the generalized Fibonacci sequence given by

$$(2) \quad F_j(x) = x^j - x^{j-1} - \dots - x - 1.$$

For studies related to the positive zeros of (2) we refer the reader to the papers by Dubeau ([D89]) and Flores ([F67]). It can be shown (see [GN99]) that for j even

$$(3) \quad F_j(x) = (x - 2 + \epsilon_j)(x + 1 - \delta_j)(x^{j-2} + a_{j-3}x^{j-3} + \dots + a_1x + a_0),$$

where $-1 + \delta_j$ and $2 - \epsilon_j$ are the negative and positive zeros of (2). Here $\{\delta_j\}$ and $\{\epsilon_j\}$ are positive, decreasing sequences. In a recent paper, Grossman and Zeleke ([GZ03]) have found an explicit form for the a_i 's in terms of ϵ_j and δ_j for $j \geq 4$. The explicit expressions for a_i as well as special cases led to some interesting identities. In this paper we present different proofs of three such identities that are hypergeometric. The paper is organized as follows. In section 2, we formulate the main results. In section 3 we provide combinatorial proofs. This requires first finding combinatorial interpretations by counting words of certain properties and defining an appropriate sign reversing involution which we call "involution". Gessel and Stanley discuss the mathematical theory related to such proofs in ([GS95]). In section 4, we present computer generated proofs of the main results. It is to be noted that there are philosophical arguments over computer-based proofs to mathematical proofs in general. It will be clear from sections 3 and 4 that the WZ method gives a unified and structured approach to proving identities of this type. Introductions to the WZ method can be found among others in the book $A = B$ ([PWZ96]) or the website <http://mathworld.wolfram.com/Wilf-ZeilbergerPair.html>. Throughout this paper we denote the set $\{k, k+1, k+2, \dots\}$ for $k \in \mathbb{Z}$ by \mathbb{N}_k .

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2. MAIN RESULTS

Theorem 1. [GZ03]

$$\sum_{k=0}^n (-1)^{n+k} \binom{n+k+1}{2k+1} 2^{2k} = n+1, \quad n \in \mathbb{N}_0.$$

Remark: The following theorems will show that the coefficients a_i 's in the expansion of positive integers are not unique.

Theorem 2.

$$\sum_{k=0}^{n+1} (-1)^{n+k+1} \binom{n+k+1}{2k} 2^{2k} = 2n+3, \quad n \in \mathbb{N}_{-1}.$$

Theorem 3. [GZ03]

$$\sum_{k=0}^{n-1} \sum_{m=2k+2}^{n+1+k} (-1)^{m+k} \binom{n+k+1}{m} 2^{m-1} = n(n+1), \quad n \in \mathbb{N}_1.$$

3. PROOFS OF THE MAIN RESULTS

3.1. Combinatorial Proofs. For a fixed $n \in \mathbb{N}_k$, $k \in \mathbb{Z}$, consider the set S of words in the alphabet $\{a, b, c\}$ such that

$$2(\text{the number of } a\text{'s}) + 1(\text{the number of } b\text{'s}) + 1(\text{the number of } c\text{'s}) = p(n),$$

where $p(n)$ is some polynomial of n . For $w \in S$, define the weight $Wt(w)$ of w by

$$Wt(w) = (-1)^{(\text{the number of } a\text{'s})}.$$

Proof of Theorem 1.

For $n \in \mathbb{N}_0$, take $p(n) = 2n+1$. Then

$$\begin{aligned} \sum_{w \in S} Wt(w) &= \sum_{k=0}^n (\text{the number of } w\text{'s with } (n-k) \text{ } a\text{'s}) Wt(w) \\ &= \sum_{k=0}^n \binom{n+k+1}{2k+1} 2^{2k+1} (-1)^{n+k} \\ &= 2 \sum_{k=0}^n (-1)^{n+k} \binom{n+k+1}{2k+1} 2^{2k}. \end{aligned}$$

Consider now $S = T \cup (S - T)$, where T is the set of all words in S of the form $c^m b^{2n+1-m}$ for $m = 0, 1, \dots, 2n+1$. Here by, for $x \in \{a, b, c\}$ the notation x^0 is used to denote the empty word. Define an “involution” as follows:

For $w \in S$ read w left to right until you either get an a , or bc . If it is an a , make it a bc . If it is a bc , make it an a . This changes the sign of $Wt(w)$ and is an involution. Note that T has $2n+2$ elements each of weight $(-1)^0 = 1$. From the involution it is clear that the sum of the weights of the elements of $S - T$ is 0. Thus $\sum_{w \in S} Wt(w) = 2n+2$. Hence the theorem follows.

Proof of Theorem 2.

Let $n \in \mathbb{N}_{-1}$ and $p(n) = 2n + 2$. Then

$$\begin{aligned} \sum_{w \in S} Wt(w) &= \sum_{k=0}^{n+1} (\text{the number of } w\text{'s with } (n+1-k) \text{ } a\text{'s}) Wt(w) \\ &= \sum_{k=0}^n \binom{n+k+1}{2k} 2^{2k} (-1)^{n+k+1} \\ &= \sum_{k=0}^n (-1)^{n+k+1} \binom{n+k+1}{2k} 2^{2k}. \end{aligned}$$

Partition S as in the proof of Theorem 1 with T the set of all words in S of the form $c^m b^{2n+2-m}$ for $m = 0, 1, \dots, 2n+2$. T has $2n+3$ elements each of weight $(-1)^0 = 1$.

Using the same “involution” as in the proof of Theorem 1 the sum of the weights of the elements of $S - T$ would be 0 and hence $\sum_{w \in S} Wt(w) = 2n+3$.

Proof of Theorem 3.

For $n \in \mathbb{N}_1$, consider the set S of words in the alphabet $\{a, b, c\}$ such that

$$1 (\text{the number of } a\text{'s}) + 1 (\text{the number of } b\text{'s}) + 1 (\text{the number of } c\text{'s}) = n+1+k,$$

for some $k \in \{0, \dots, n-1\}$ and (the number of b 's) + (the number of c 's) is at least $2k+2$. For $w \in S$, define the weight $Wt(w)$ of w by

$$Wt(w) = (-1)^{(\text{the number of } a\text{'s})}.$$

Then

$$\begin{aligned} \sum_{w \in S} Wt(w) &= \sum_{k=0}^{n-1} \sum_{m=2k+2}^{n+k+1} (\text{the number of } w\text{'s with } (n+1+k-m) \text{ } a\text{'s}) Wt(w) \\ &= \sum_{k=0}^{n-1} \sum_{m=2k+2}^{n+k+1} \binom{n+k+1}{m} 2^m (-1)^{n+1+m+k} \\ &= 2(-1)^{n+1} \sum_{k=0}^{n-1} \sum_{m=2k+2}^{n+k+1} \binom{n+k+1}{m} 2^{m-1} (-1)^{m+k}. \end{aligned}$$

Read a word $w \in S$ from left to right. Count the number b and c until the sum is 3. Thus w has the form $a^l x a^n y a^m z *$ where $l, m, n \in \mathbb{N}_0$ and $x, y, z \in \{b, c\}$. For such words, define a mapping σ as follows:

$$\sigma(w) = \begin{cases} a^l x a^{n+1} y a^m z * & : \text{ if } n, m \text{ have same parity and } n \neq 1, \\ a^l x a^{n-1} y a^m z * & : \text{ if } n, m \text{ have same parity and } n = 1, \\ a^l x a^{n-1} y a^m z * & : \text{ if } n, m \text{ have opposite parity and } n \neq 0, \\ a^l x a^{n+1} y a^m z * & : \text{ if } n, m \text{ have opposite parity and } n = 0. \end{cases}$$

Clearly σ is an “involution”. This involution is not defined for elements of S of length $n+1$ and the number of b 's and c 's exactly 2. There are $4 \binom{n+1}{2} = 2n(n+1)$ such words each of weight $(-1)^{n+1}$ and hence the theorem follows.

3.2. The WZ Method Proofs. Proof of Theorem 1.

Let $F(n, k) = \binom{n+k+1}{2k+1} \frac{2^{2k} (-1)^{k+n+1}}{n+1}$ and let $S(n) = \sum_{k=0}^n F(n, k)$. We want to show that $S(n) = 1$ for all $n \in \mathbb{N}_0$. F satisfies the recurrence equation:¹

$$(4) \quad F(n+1, k) + F(n, k) = G(n, k+1) - G(n, k),$$

where $G(n, k) = R(n, k) F(n, k)$ and $R(n, k) = -\frac{k(2k+1)}{(n+1-k)(n+2)}$.

By summing both sides of equation (4) with respect to k we get $S(n+1) - S(n) = 0$. Moreover, $S(0) = 1$ and hence $S(n) = 1$ for all $n \in \mathbb{N}_0$.

Proof of Theorem 2.

Let $F(n, k) = \binom{n+k+1}{2k} \frac{2^{2k} (-1)^{k+n+1}}{2n+3}$ and let $S(n) = \sum_{k=0}^{n+1} F(n, k)$. We want to show that $S(n) = 1$ for all $n \in \mathbb{N}_{-1}$. F satisfies the recurrence equation:¹

$$(5) \quad F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k),$$

where $G(n, k) = R(n, k) F(n, k)$ and $R(n, k) = \frac{2k(2k-1)}{(n+2-k)(2n+5)}$.

By summing both sides of equation (5) with respect to k we get $S(n+1) - S(n) = 0$. Moreover, $S(-1) = 1$ and hence $S(n) = 1$ for all $n \in \mathbb{N}_{-1}$.

Proof of Theorem 3.

Reversing the order of summation, the identity can be rewritten as

$$(6) \quad \sum_{m=2}^{2n} \sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{n+k+1}{m} 2^{m-1} (-1)^{m+k+n+1} = n(n+1).$$

Let us denote the left side of (6) by $S(n)$ and its summand by $F(n, k, m)$, i.e.

$$F(n, k, m) = \binom{n+k+1}{m} 2^{m-1} (-1)^{m+k+n+1}.$$

Then F satisfies the recurrence equation:²

$$(7) \quad F(n+1, k, m) - F(n, k, m) = F(n, k+1, m) - F(n, k, m).$$

Summing both sides of (7) with respect to k and with respect to m , we get

$$(8) \quad \begin{aligned} & S(n+1) - S(n) \\ &= \sum_{m=2}^{2n} \binom{n + \lfloor m/2 \rfloor + 1}{m} 2^{m-1} (-1)^{m + \lfloor m/2 \rfloor + n + 1} - \sum_{m=2}^{2n} \binom{n+1}{m} 2^{m-1} (-1)^{m+n+1}. \end{aligned}$$

¹The recurrence equation is automatically generated by a MAPLE package EKHAD which is available from <http://www.math.rutgers.edu/~zeilberg/>

²The recurrence equation is automatically generated by MultiSum, a Mathematica package which is available from <http://www.risc.uni-linz.ac.at/research/combinat/risc/software/>

But

$$\begin{aligned}
 \sum_{m=2}^{2n} \binom{n+1}{m} 2^{m-1} (-1)^{m+n+1} &= \frac{(-1)^{n+1}}{2} \sum_{m=0}^{n+1} \binom{n+1}{m} (-2)^m - (n+1) (-1)^n + \frac{(-1)^n}{2} \\
 &= \frac{(-1)^{n+1}}{2} (1-2)^{n+1} - (n+1) (-1)^n + \frac{(-1)^n}{2} \\
 (9) \qquad \qquad \qquad &= \frac{1+(-1)^n}{2} - (n+1) (-1)^n,
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{m=2}^{2n} \binom{n+\lfloor m/2 \rfloor + 1}{m} 2^{m-1} (-1)^{m+\lfloor m/2 \rfloor + n+1} \\
 &= \sum_{m=1}^n \binom{n+m+1}{2m} 2^{2m-1} (-1)^{m+n+1} + \sum_{m=1}^{n-1} \binom{n+m+1}{2m+1} 2^{2m} (-1)^{m+n} \\
 &= \sum_{m=0}^{n+1} \binom{n+m+1}{2m} 2^{2m-1} (-1)^{m+n+1} - 2^{2n+1} + \frac{(-1)^n}{2} \\
 &\quad + \sum_{m=0}^n \binom{n+m+1}{2m+1} 2^{2m} (-1)^{m+n} - (n+1) (-1)^n - 2^{2n} \\
 (10) \qquad \qquad \qquad &= n + \frac{3}{2} - 2^{2n+1} + \frac{(-1)^n}{2} + (n+1) - (n+1) (-1)^n - 2^{2n}.
 \end{aligned}$$

From equations (8), (9) and (10), we get $S(n+1) - S(n) = 2(n+1)$. Since $S(1) = 2$, and $n(n+1)$ satisfies the same recurrence relation, therefore $S(n) = n(n+1)$ for all $n \in \mathbb{N}_1$.

SOME COROLLARIES.

For completeness, we state the following results from ([GZ03]) and prove using theorems 1-3.

Corollary 1.

$$\sum_{k=0}^n \sum_{m=2k+1}^{n+k+1} (-1)^{m+k+n+1} \binom{n+k+1}{m} 2^{m-1} = (n+1)^2.$$

Proof: The result follows by adding theorems 1 and 3.

Corollary 2.

$$\sum_{k=0}^{2n} \sum_{m=2k+2}^{2n+2k+2} (-1)^{m+k} \binom{2n+k+2}{m} 2^{m-1} = (2n+1)(2n+2).$$

Proof: Add theorem 3 and 1 and multiply the result by 2.

Corollary 3.

$$\sum_{k=0}^{2n+1} \sum_{m=2k+1}^{2n+k+2} (-1)^{m+k} \binom{2n+k+2}{m} 2^{m-1} = (2n+2)^2.$$

Proof: Replace n by $2n+1$ in theorem 1.

Corollary 4.

$$\sum_{k=0}^{2n} \sum_{m=2k+1}^{2n+k+2} (-1)^{m+k+1} \binom{2n+k+1}{m} 2^{m-1} = (2n+1)^2 .$$

Proof: Replace n by $2n$ in theorem 1.

Corollary 5.

$$\sum_{k=0}^{2l} \sum_{m=1}^{2k+2} (-1)^m \binom{2k+m+1}{2m-1} 4^{m-1} = (2l+1)(2l+2) .$$

Proof: Replace n by $2n+1$ in theorem 1 and sum k from 0 to $2l$.

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DEPARTMENT OF MATHEMATICS, CENTRAL MICHIGAN UNIVERSITY, MT. PLEASANT, MI 48859, GROSS1GW@CMICH.EDU

DEPARTMENT OF MATHEMATICS, GRAND VALLEY STATE UNIVERSITY, ALLENDALE , MI 49401, TEFERAA@GVSU.EDU

DEPARTMENT OF MATHEMATICS, ALMA COLLEGE, ALMA, MI 48801, ZELEKE@ALMA.EDU